# Lecture 6. Direction Fields and the Existence and Uniqueness Theorem

### More general differential equations

- Now we consider differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

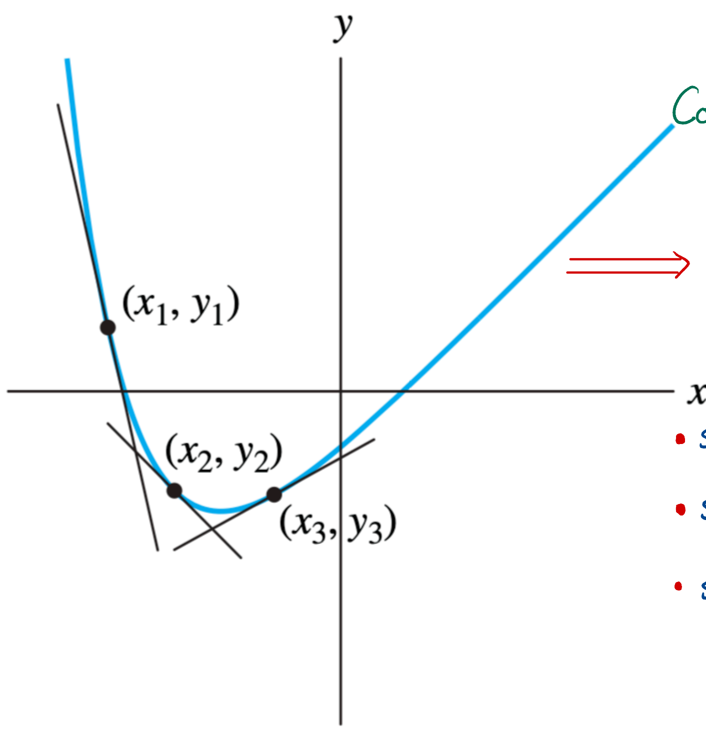
where the right-hand function $f(x, y)$ involves both the independent variable $x$ and the dependent variable $y$.

- Using *graphical* and *numerical* methods we can construct approximate solutions of differential equations.

### Slope Fields and Graphical Solutions

Differential Equations and Slopes

- We can view solutions of the differential equation $y' = f(x, y)$ in a simple geometric way.
- It rests on the general fact that *first derivatives give slopes of tangent lines*.
- Thus at each point $(x, y)$ of the $xy$ -plane, the value of $f(x, y)$ determines a slope $m = f(x, y)$
- A **solution curve** of the differential equation is then simply a curve in the $xy$ -plane whose tangent line at $(x, y)$ has slope $f(x, y)$




Consider the diff. eqn
$y' = x - y$

This is a solution curve for the eqn. $y' = x - y$ together with the tangent lines having

- slope $m_1 = x_1 - y_1$ at the point $(x_1, y_1)$
- slope $m_2 = x_2 - y_2$ at the point $(x_2, y_2)$
- slope $m_3 = x_3 - y_3$ at the point $(x_3, y_3)$

**Graphical Method**

- This leads to a graphical method for constructing approximate solutions of the differential equation

$$\frac{dy}{dx} = f(x,y)$$

- First we choose a representative collection of points $(x,y)$ in the plane.
- Through each point $(x,y)$ we draw a short line segment having slope $m = f(x,y)$
- All these line segments taken together constitute a **slope field** for the equation $y' = f(x,y)$

also called direction field

| $x \setminus y$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|---|---|
| $-4$ | 0 | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ | $-8$ |
| $-3$ | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ |
| $-2$ | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ |
| $-1$ | 3 | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ |
| $0$ | 4 | 3 | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ |
| $1$ | 5 | 4 | 3 | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ |
| $2$ | 6 | 5 | 4 | 3 | 2 | 1 | 0 | $-1$ | $-2$ |
| $3$ | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | $-1$ |
| $4$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

**Figure** Values of the slope $y' = x - y$ for $-4 \le x, y \le 4$

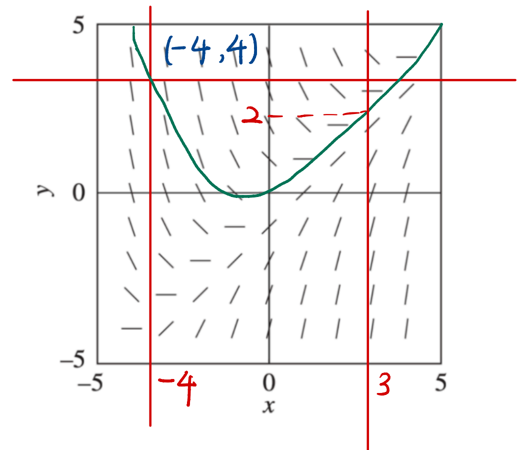



**Figure** Slope field for $y' = x - y$ for the table of slopes

**Question**: Let $y(x)$ be the solution of the initial value problem $y' = x - y, y(-4) = 4.$

f(x,y)

Can you estimate the value of $y(3)$? $\approx 2$

- We first choose a collection of pts $(x,y)$ in the plane and calculate the slope $y' = x - y$ at each points
- For example, we take $x, y$ from $-4, -3, \cdots, 3, 4$
- Then at some point, say $(1,-3)$, the slope is

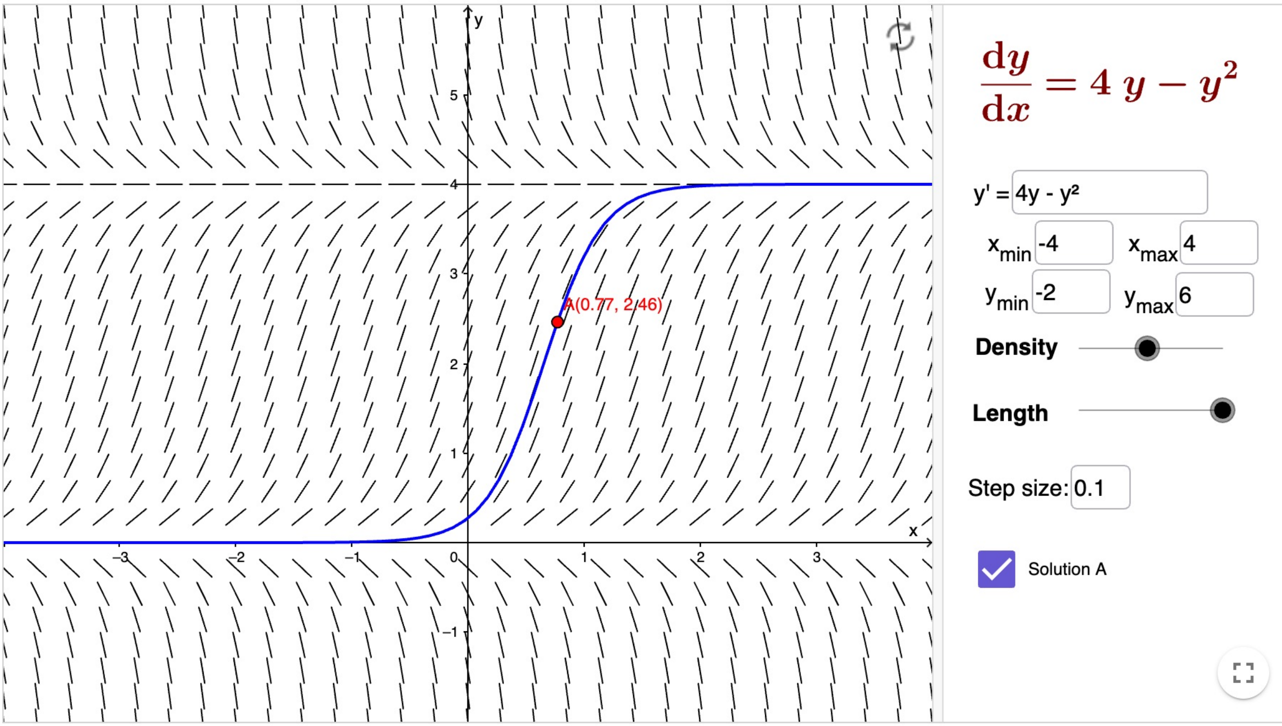
$$\frac{dy}{dx} = x - y = 1-(-3) = 4$$

- At each point, we draw a short line segment with the slope we calculated.

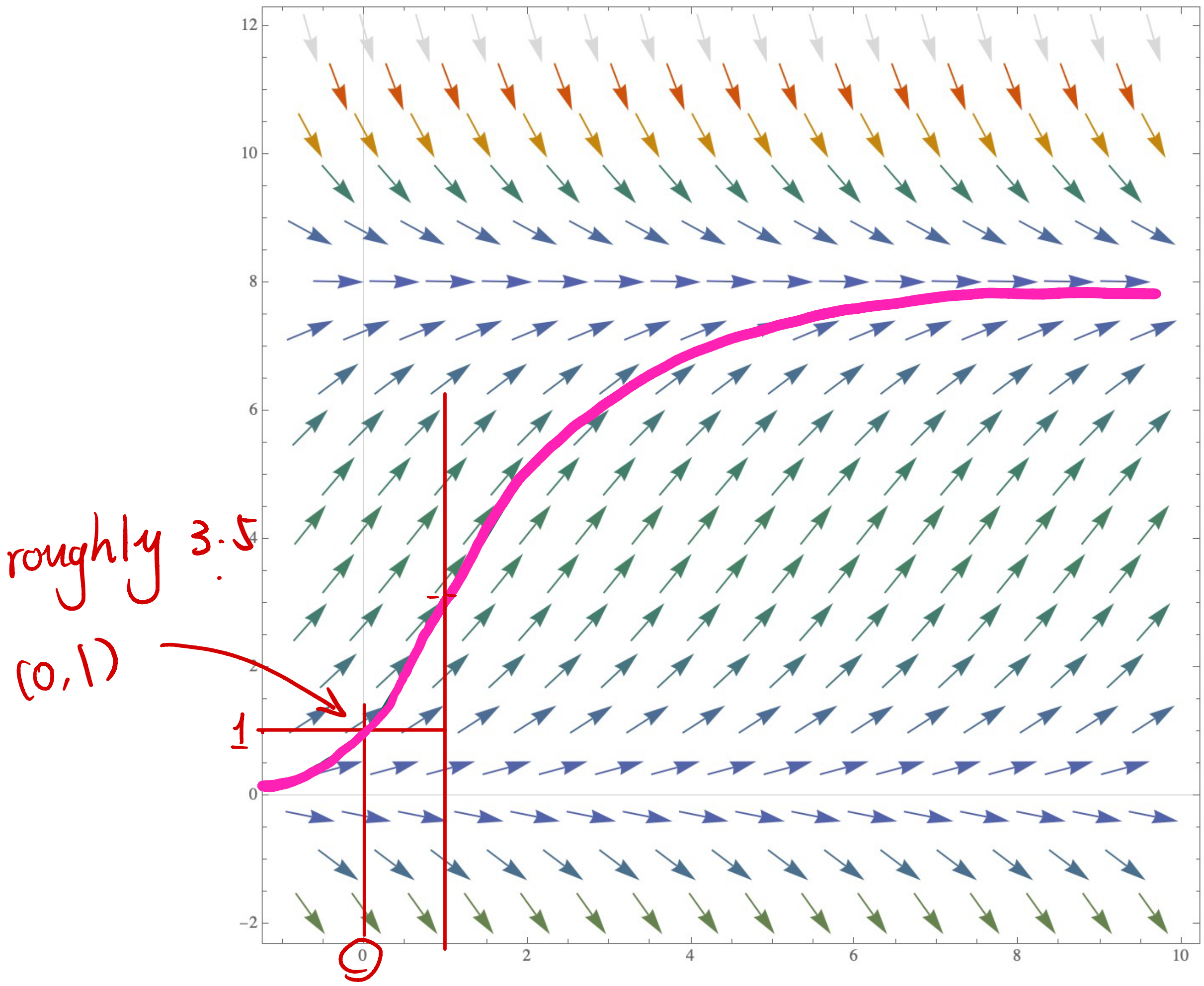
An online slope field generator:

https://www.geogebra.org/m/Pd4Hn4BR

This applet will generate Direction Fields and approximate solution curves given initial values.
**Click and drag** the initial point A to see its corresponding solution curve  Credits:  Originally created by Chip Rollinson



**Example 1** Consider the direction field of some differential equation $\frac{dy}{dt} = F(t, y)$.




Suppose that $y(0) = 1$. Then $y(1)$ is closest to which value?

**A.** $1.5$

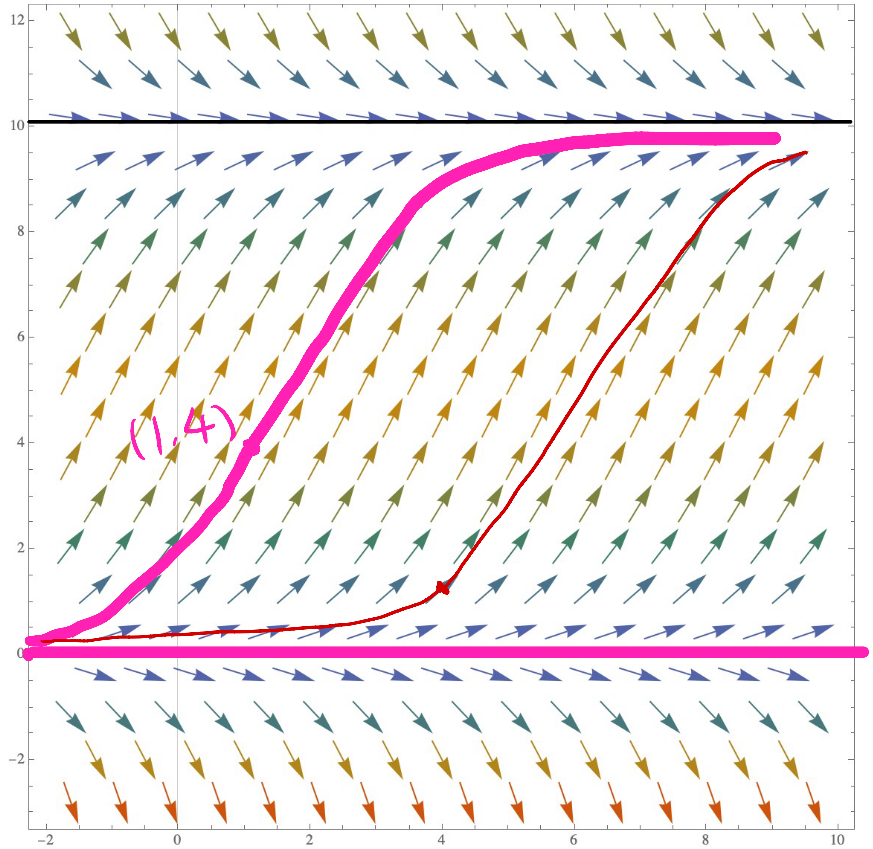
**B.** $3.5$

**C.** $-1.5$

**D.** $0$

**E.** $-2$

**Example 2.** The slope field for the equation $dP/dt = 0.1P(10 - P)$, for $P \geq 0$, is shown below.




On a print out of this slope field, sketch the solutions that pass through $(0, 0)$; $(1, 4)$; $(4, 1)$; $(-4.5, 1)$; $(-2, 12)$; and $(-2, 10)$.

(1) For which positive values of $P$ are the solutions increasing? (Give your answer as an interval or list of intervals)

$(0, 10)$

(2) For what positive values of $P$ are the solutions decreasing?

$(10, \infty)$

(3) What is the equation of the solution to this differential equation that passes through $(0, 0)$?

$P(t) \equiv 0$

(4) If the solution passes through a value of $P > 0$, what is the limiting value of $P$ as $t$ gets large?

$\lim_{t \to \infty} P(t) = 10$

A constant solution of a differentail equation is sometimes called an **equilibrium solution**.

For example,

$$\frac{dx}{dt} = 4x(7-x) \tag{1}$$

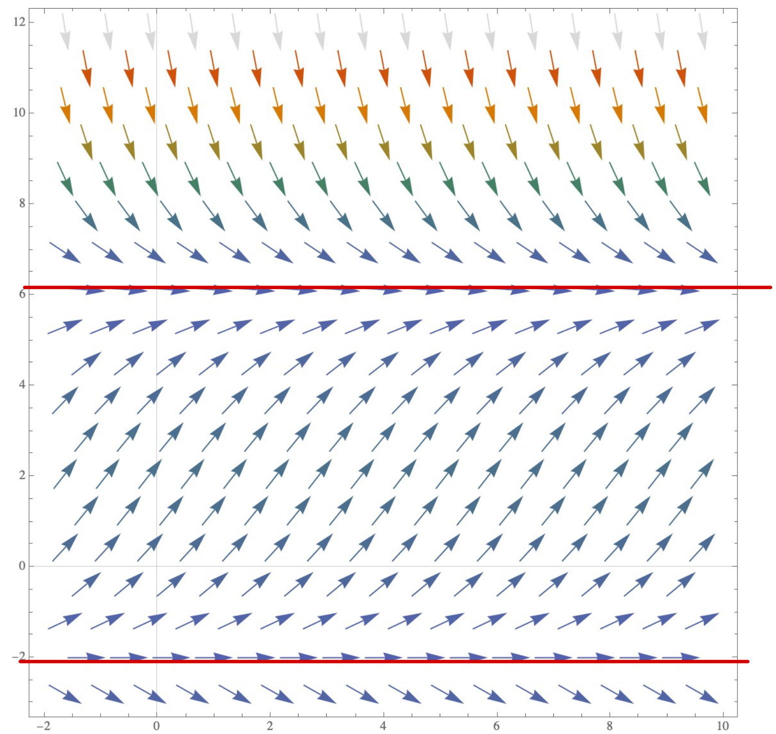
Then the constant functions $x(t) \equiv 0$ and $x(t) \equiv 7$ are the equilibrium solutions to Eq (1).

We will discuss more about them in the next lecture.

It is very easy to identify the equilibrium solutions via the direction field. For the equation (1), the equilibrium solutions correponsponds to the curves with slope $0$. For example,

**Example 3.** Consider the direction field below for a differential equation.

Use the graph to find the equilibrium solutions.

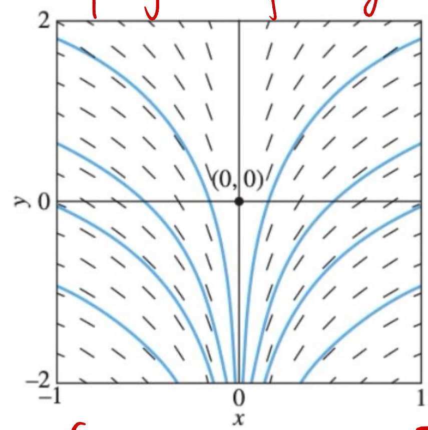



$x(t) \equiv -2,\ 6$

## Existence and Uniqueness of Solutions

**Question 1** (Note the exisitence of solutions fails for this question)

Slope field for $y' = \frac{1}{x}$




Does the following initial value problem have a solution?

$$y' = \frac{1}{x},\ y(0) = 0$$

**Solution**. We have

$$\frac{dy}{dx} = \frac{1}{x}$$

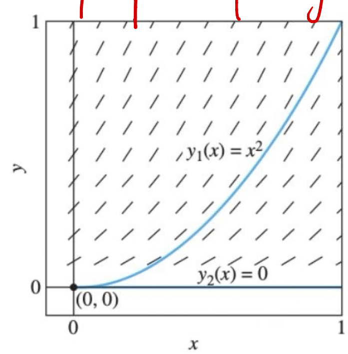
$$\Rightarrow \quad y = \ln|x| + c$$

From the slope field, **no solution** curve pass through point $(0, 0)$

Note $f(x,y) = \frac{1}{x}$ is not continuous at $x=0$ (compare with Thm1)

**Question 2** (Note the uniqueness of solutions fails for this question)

Slope field for $y' = 2\sqrt{y}$




Verify $y_1 = x^2$ and $y_2 \equiv 0$ satisfy the following initial value problem.

$$y' = 2\sqrt{y},\ y(0) = 0$$

**Solution**.

If $y_1 = x^2$, then $y_1' = 2x = 2\sqrt{y}$, and $y_1(0) = 0$

If $y_2 \equiv 0$, then $y_2' = 0 = 2 \cdot y_2$ and $y_2(0) = 0$

So we have at least 2 solutions satisfy the equation with the initial condition. Thus solution for the initial value problem is **not unique**.

Note $f(x,y) = 2\sqrt{y}$, and $\frac{\partial f}{\partial y}$ is not continuous is not continuous at point $(0,0)$. (compare with Thm1)

🤔 **When we have both existence and uniqueness for the initial value problem?**

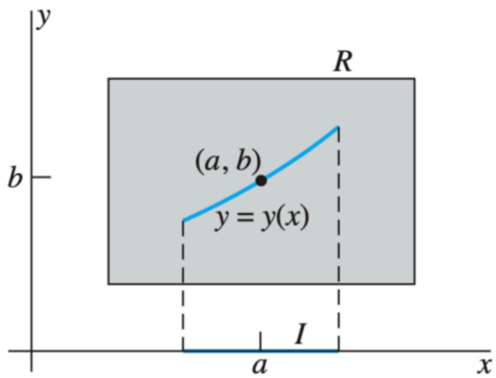
$\frac{\partial f}{\partial y}$

**THEOREM 1 Existence and Uniqueness of Solutions**

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle $R$ in the $xy$ -plane that contains the point $(a, b)$ in its interior. Then, for some open interval $I$ containing the point $a$, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval $I$. (As illustrated in Fig. 1 the solution interval $I$ may not be as "wide" as the original rectangle $R$ of continuity. See the **Remark** below)




**Remark:** If the hypotheses of **Theorem 1** are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.

**Exercise 4.** Suppose $\frac{dy}{dx} = f(x, y) = x \ln y; \quad y(1) = 1$

(a) $\frac{\partial f}{\partial y} =$ $x \cdot \frac{1}{y}$

(b) Since the function $f(x, y)$ is continuous at the point $(1, 1)$, the partial derivative exists and is continuous near the point $(1, 1)$, the solution to $\frac{dy}{dx} = f(x, y)$ exists and it is unique near $y(1) = 1$.

**Exercise 5.** Consider the first order differential equation

$$y' + \frac{t}{t^2 - 25} y = \frac{e^t}{t - 7}.$$

For each of the initial conditions below, determine the largest interval $a < t < b$ on which the existence and uniqueness theorem for first order linear differential equations guarantees the existence of a unique solution.

(1) $y(-7) = -5.5$ (2) $y(-2.5) = -2.1$ (3) $y(0) = 0$

(4) $y(5.5) = 1.7$ (5) $y(8) = -0.5$

ANS: We first write the given eqn in the form of

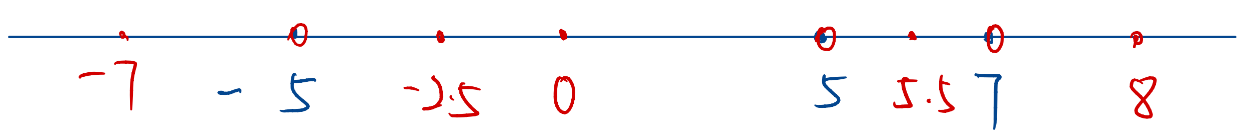
$$y' = f(t, y),$$

So we have $y' = -\frac{t}{t^2-25} y + \frac{e^t}{t-7}$.

Thus $f(t,y) = -\frac{t}{t^2-25} y + \frac{e^t}{t-7}$.

Then $\frac{\partial f}{\partial y} = -\frac{t}{t^2-25}$.

Therefore $f$ and $\frac{\partial f}{\partial y}$ are continuous except at $t = \pm 5$ $t = 7$.




Thus (1) $t < -5$ as $-7 \in (-\infty, -5)$

(2) $-5 < t < 5$ as $-2.5 \in (-5, 5)$

(3) $-5 < t < 5$ as $0 \in (-5, 5)$

(4) $5 < t < 7$ as $5.5 \in (5, 7)$

(5) $t > 7$ as $8 \in (7, \infty)$